# MAT124 MATHEMATICS II

Vector Calculus (cont.), Green's Theorem in the Plane

# Outline

Vector Calculus (cont.)

Examples: Interpreting Divergence and Curl

Green's Theorem in the Plane

# Vector Calculus (cont.)

# Some Identities Involving Grad, Div, and Curl

Theorem. *Vector differential identities*

Let $\phi$ and $\psi$ be scalar fields and $\mathbf{F}$ and $\mathbf{G}$ be vector fields, all assumed sufficiently smooth. Then:

(a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(b) $\nabla \cdot (\phi\mathbf{F}) = (\nabla\phi) \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F})$

(c) $\nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$

(d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(e) $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

(f) $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$

(g) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ (div curl $= 0$)

(h) $\nabla \times (\nabla\phi) = 0$ (curl grad $= 0$)

(i) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$ (curl curl $=$ grad div $-$ Laplacian).

(Here $\nabla^2 = \nabla \cdot \nabla$, called the Laplacian operator.)

# Gradient, Divergence, and Curl

## Scalar and Vector Potentials

Two special terms are used to describe vector fields for which either the divergence or the curl is identically zero.

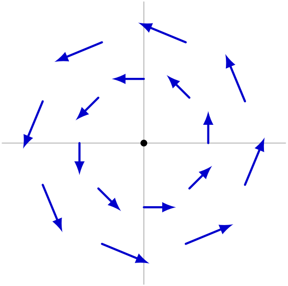
### Solenoidal and Irrotational Vector Fields

A vector field $\mathbf{F}$ is called **solenoidal** in a domain $D$ if $\operatorname{div}\mathbf{F} = 0$ in $D$.

A vector field $\mathbf{F}$ is called **irrotational** in a domain $D$ if $\operatorname{curl}\mathbf{F} = 0$ in $D$.

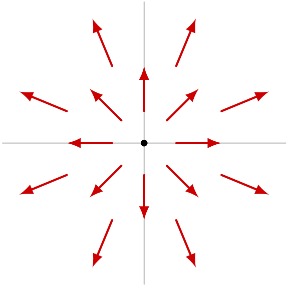
**Example of Solenoidal Field**

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j} \qquad (\operatorname{div}\mathbf{F} = 0)$$



**Example of Irrotational Field**

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} \qquad (\operatorname{curl}\mathbf{F} = \mathbf{0})$$



# Gradient, Divergence, and Curl

## Scalar and Vector Potentials

- Every conservative vector field is irrotational.
- The curl of any vector field is solenoidal.

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The converses of these assertions hold if the domain of $\mathbf{F}$ satisfies certain conditions.

**Theorem.**

If $\mathbf{F}$ is a smooth, irrotational vector field on a simply connected domain $D$, then $\mathbf{F} = \nabla\phi$ for some scalar potential function defined on $D$, so $\mathbf{F}$ is conservative.

# Gradient, Divergence, and Curl

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**Theorem.**

If $\mathbf{F}$ is a smooth, solenoidal vector field on a domain $D$ with the property that every closed surface in $D$ bounds a domain contained in $D$, then $\mathbf{F} = \operatorname{curl}\mathbf{G}$ for some vector field $\mathbf{G}$ defined on $D$. Such a vector field $\mathbf{G}$ is called a vector potential of $\mathbf{F}$.

# Gradient, Divergence, and Curl

## Scalar and Vector Potentials

**EXAMPLE**

Show that $\mathbf{F} = (x^2 + yz)\,\mathbf{i} - 2y(x+z)\,\mathbf{j} + (xy + z^2)\,\mathbf{k}$ is solenoidal in $\mathbb{R}^3$ and find a vector potential.

# Gradient, Divergence, and Curl

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**Solution:** Since $\operatorname{div}\mathbf{F} = 2x - 2(x+z) + 2z = 0$ in $\mathbb{R}^3$, $\mathbf{F}$ is solenoidal. A vector potential $\mathbf{G}$ for $\mathbf{F}$ must satisfy $\operatorname{curl}\mathbf{G} = \mathbf{F}$; that is,

$$\begin{aligned}
\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} &= x^2 + yz, \\
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Let us try to find a solution $\mathbf{G}$ with $G_2 = 0$ identically.

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Let us try to find a solution $\mathbf{G}$ with $G_2 = 0$ identically.

$$G_3 = \int (x^2 + yz)\,dy = x^2 y + \frac{1}{2}y^2 z + M(x,z).$$

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Now we make a second simplifying assumption, that $M(x,z) = 0$.

# Gradient, Divergence, and Curl

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$$\frac{\partial G_1}{\partial z} = \frac{\partial G_3}{\partial x} - 2xy - 2yz = 2xy - 2xy - 2yz = -2yz.$$

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## Scalar and Vector Potentials

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$$G_1 = -2\int yz\,dz = -yz^2 + N(x,y).$$

# Gradient, Divergence, and Curl

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$$xy + z^2 = -\frac{\partial G_1}{\partial y} = z^2 - \frac{\partial N}{\partial y} \quad \text{Hence} \quad \frac{\partial N}{\partial y} = -xy.$$

# Gradient, Divergence, and Curl

## Scalar and Vector Potentials

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Thus, $\frac{\partial N}{\partial y} = -xy$; observe that the terms involving $z$ have cancelled out. This happened because $\operatorname{div} \mathbf{F} = 0$. Had $\mathbf{F}$ not been solenoidal, we could not have determined $N$ as a function of $x$ and $y$ only from the above equation. As it is, however,

$$N(x,y) = -\int xy\,dy = -\frac{1}{2}xy^2 + P(x).$$

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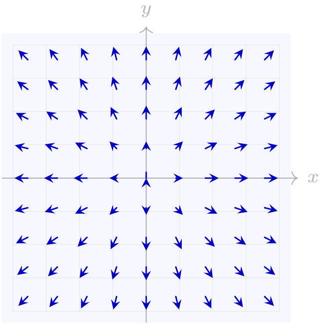
Choose $P(x) = 0$. A valid vector potential is

$$\boxed{\mathbf{G} = -\left(yz^2 + \frac{xy^2}{2}\right)\mathbf{i} + \left(x^2y + \frac{y^2z}{2}\right)\mathbf{k}.}$$

# Examples: Interpreting Divergence and Curl

# Gradient, Divergence, and Curl

## Example Gallery — Pure Source



$\mathbf{F}(x, y) = \langle x, y\rangle$

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# Gradient, Divergence, and Curl

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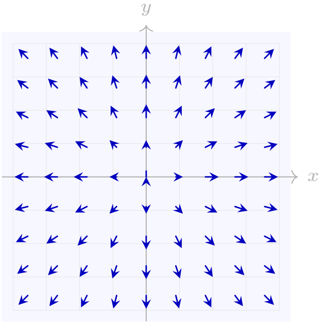

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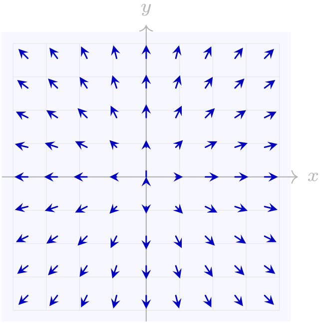

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$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 - 0 = 0.$$

# Gradient, Divergence, and Curl

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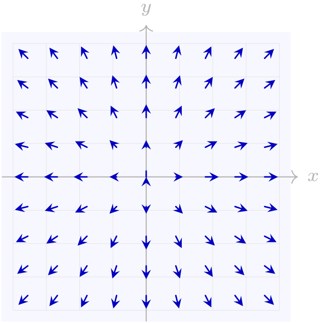

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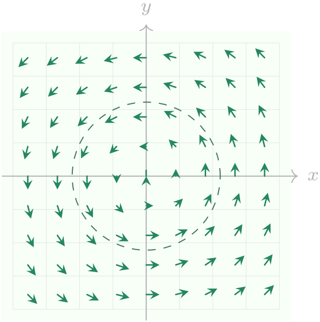
**Interpretation**

The field spreads outward from the origin. It has positive divergence, but no local rotation.

# Gradient, Divergence, and Curl

## Example Gallery — Pure Rotation



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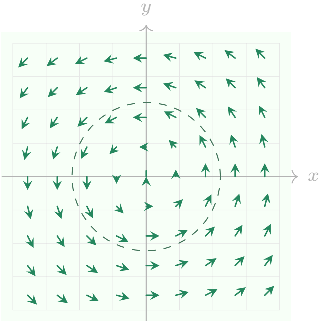

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# Gradient, Divergence, and Curl

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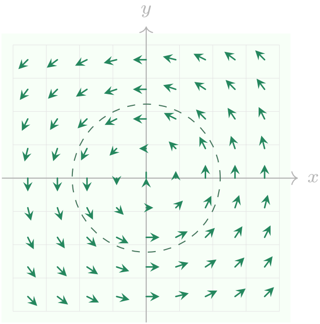

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$$\begin{aligned}(\nabla \times \mathbf{F}) \cdot \mathbf{k} &= \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \\ &= 1 - (-1) = 2.\end{aligned}$$

# Gradient, Divergence, and Curl

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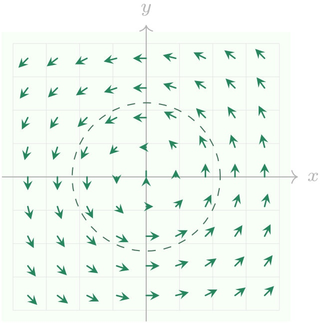

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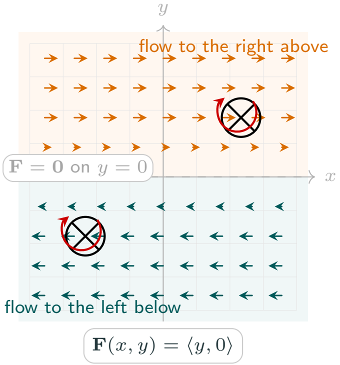
$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)$$
$$= 1 - (-1) = 2.$$

**Interpretation**

The field rotates counterclockwise. There is no net expansion or compression, but the curl is positive.

# Gradient, Divergence, and Curl

## Example Gallery — Shear Flow



$$\mathbf{F}(x, y) = \langle y, 0\rangle.$$

# Gradient, Divergence, and Curl

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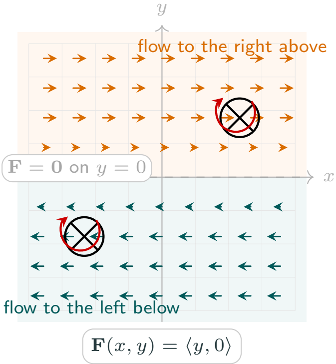

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# Gradient, Divergence, and Curl

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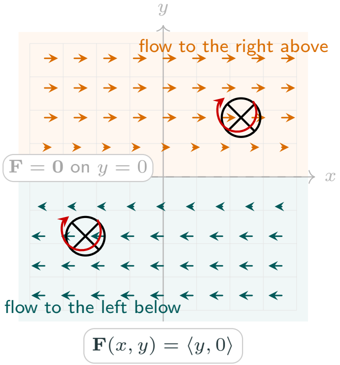

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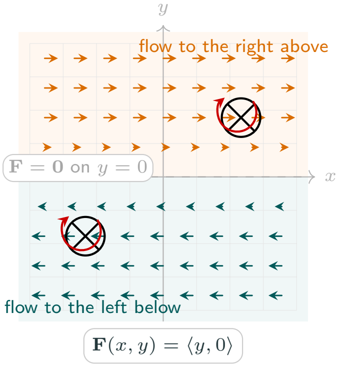

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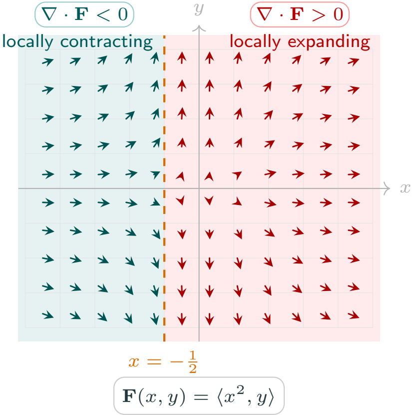
**Interpretation**

The streamlines are straight, but the field still has nonzero curl. The flow above moves to the right, while the flow below moves to the left, so a tiny paddle wheel rotates clockwise.

# Gradient, Divergence, and Curl

## Example Gallery — Nonuniform Expansion



$$\mathbf{F}(x, y) = \langle x^2, y \rangle.$$

# Gradient, Divergence, and Curl

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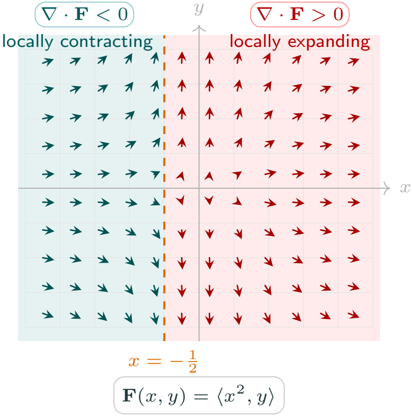

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# Gradient, Divergence, and Curl

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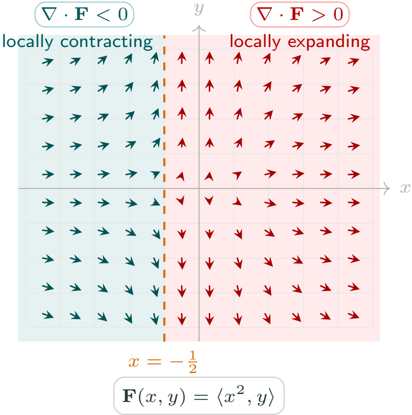

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# Gradient, Divergence, and Curl

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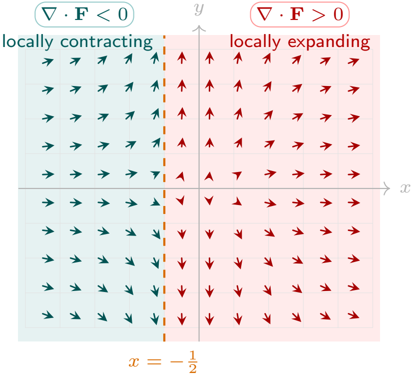

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$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x^2) = 0.$$

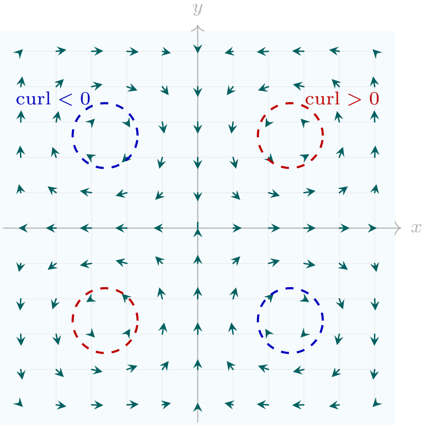
### Interpretation

There is no local rotation, so the curl is zero everywhere.

The field behaves like a local sink on the left and a local source on the right of $x = -\frac{1}{2}$.

# Gradient, Divergence, and Curl

## Example Gallery — Alternating Vortices



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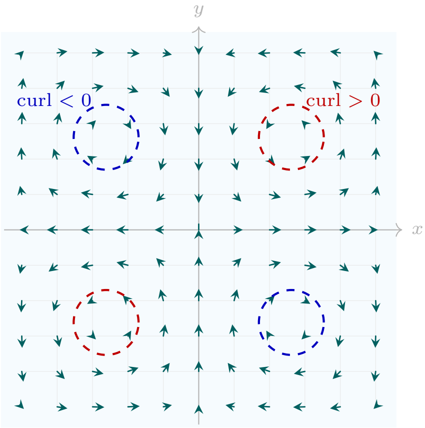

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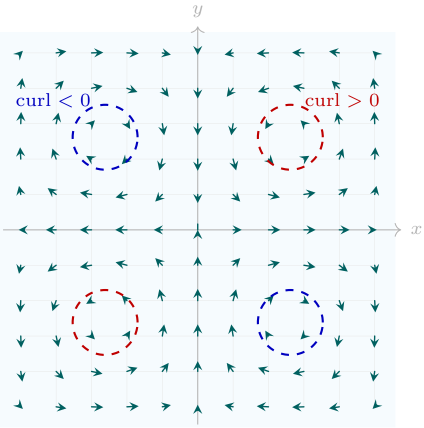

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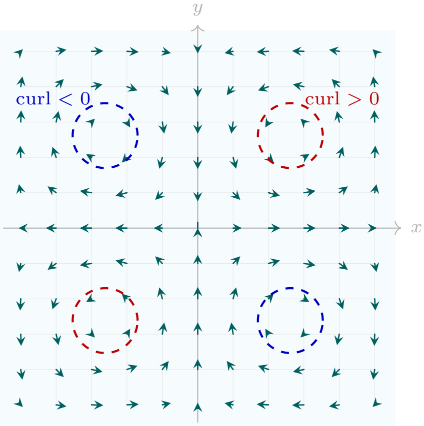

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**Interpretation**

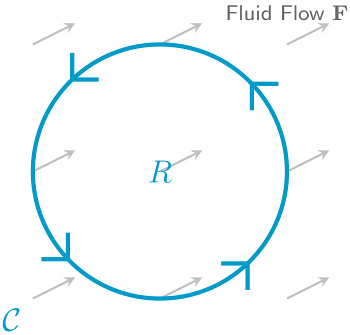
The field is divergence-free, but the local rotation changes sign from region to region. Some cells rotate counterclockwise, while others rotate clockwise.

# Green’s Theorem in the Plane

# Green's Theorem

## The Rowboat Analogy

Here's a great way to think about the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$: **Imagine rowing a tiny boat around the loop $C$ counter-clockwise.**



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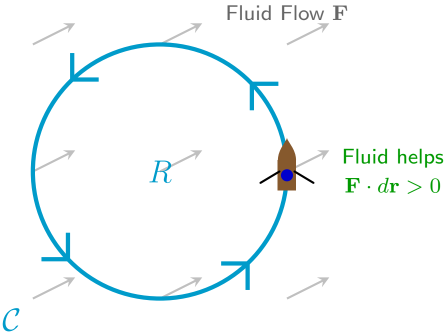

At each point of your journey, the vector field $\mathbf{F}$ acts as the fluid's current.

If the current **pushes you forward**, the dot product is positive (it was helpful).

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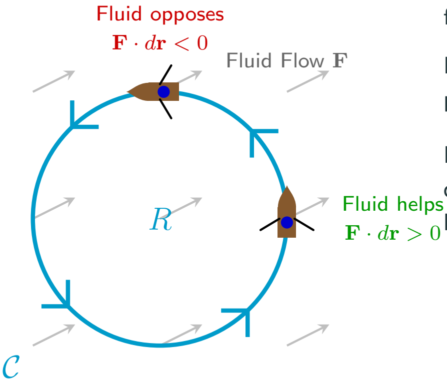

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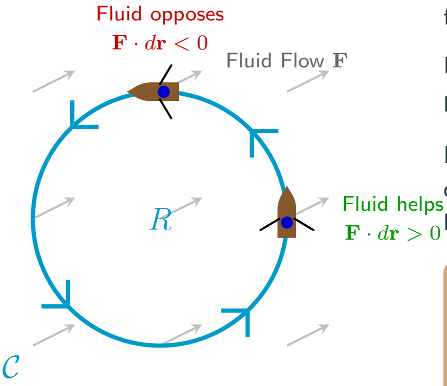

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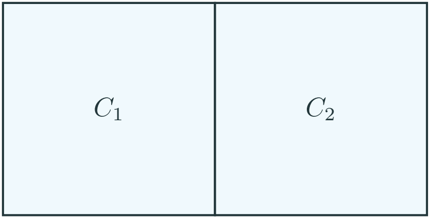
**Net Circulation**

The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ simply adds up all these dot products to tell you if the flow was generally helpful ($> 0$) or burdensome ($< 0$) overall.

# Green's Theorem

## The Cancellation Principle

Now, what happens if we place **two tiny rowboats** next to each other, both rowing counter-clockwise?



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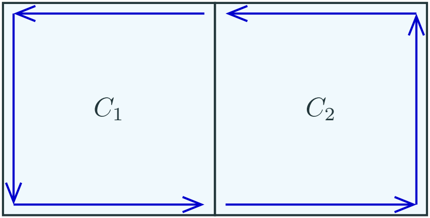

The circulation around a tiny loop is driven by its local **curl**. If we sum the circulations of both regions:

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} \approx \sum_{i=1}^{2} (\operatorname{curl} \mathbf{F}\cdot\mathbf{k})_i \Delta A$$

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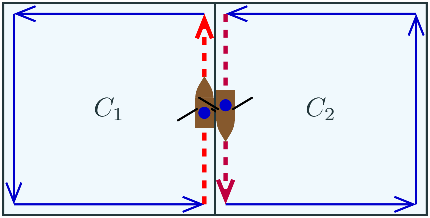

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Notice the shared boundary. One boat rows **UP**, while the other rows **DOWN**.

Since they are in the exact same current but moving in opposite directions, the fluid helps one exactly as much as it opposes the other.

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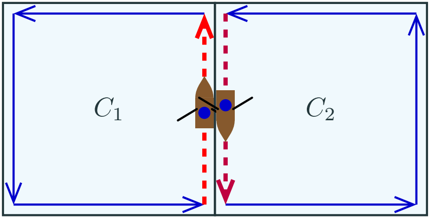

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Notice the shared boundary. One boat rows **UP**, while the other rows **DOWN**.

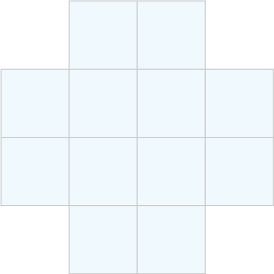
Since they are in the exact same current but moving in opposite directions, the fluid helps one exactly as much as it opposes the other.

They perfectly **cancel each other out!** The net result is simply the circulation along the **outer boundary**.

# Green's Theorem

## From Microscopic to Macroscopic

Imagine dividing a large macroscopic region $R$ into a grid of tiny pieces, each with its own tiny rowboat.



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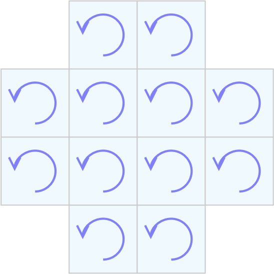

The circulation of a single tiny boat $i$ is its local 2D-curl multiplied by its tiny area:

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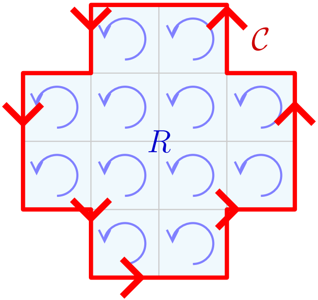

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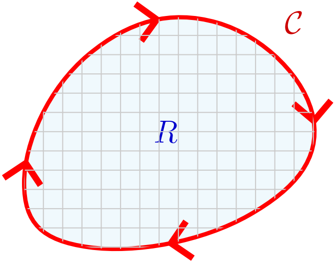
If we sum the efforts of all the tiny boats, **all internal rowing cancels out**. Only the boats on the outer boundary $\mathcal{C}$ have no neighbor to cancel with!

$$\sum_i \oint_{C_i} \mathbf{F} \cdot d\mathbf{r} = \oint_{\text{Outer}} \mathbf{F} \cdot d\mathbf{r}$$

# Green’s Theorem

## Arbitrary Curved Regions

What if our region is not a perfect grid? We can approximate any curved region $R$ with a fine grid of tiny rectangular boxes.



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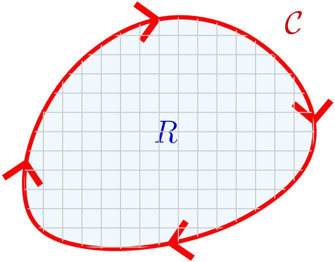

As we make the boxes smaller and smaller ($\Delta A \to 0$):

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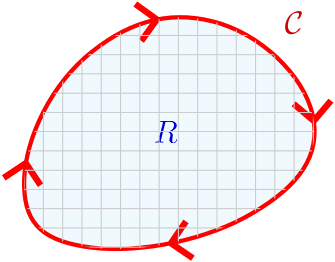

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# Green's Theorem in the Plane

Theorem. *Green's Theorem*

Let $R$ be a regular, closed region in the $xy$-plane whose boundary, $\mathcal{C}$, consists of one or more piecewise smooth, simple closed curves that are positively oriented with respect to $R$. If

$$\mathbf{F} = F_1(x,y)\,\mathbf{i} + F_2(x,y)\,\mathbf{j}$$

is a smooth vector field on $R$, then

$$\oint_{\mathcal{C}} F_1(x,y)\,dx + F_2(x,y)\,dy = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

# Green's Theorem in the Plane

Green's Theorem is a two-dimensional version of the Fundamental Theorem of Calculus that expresses the double integral of a certain kind of derivative of a two-dimensional vector field $\mathbf{F}(x, y)$, namely, the $\mathbf{k}$-component of $\operatorname{curl}\mathbf{F}$, over a region $R$ in the $xy$-plane as a line integral of the tangential component of $\mathbf{F}$ around the curve $\mathcal{C}$ which is the oriented boundary of $R$:

$$\iint_R \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA = \oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}.$$

# Green's Theorem in the Plane

**EXAMPLE (Area bounded by a simple closed curve)**
For any of the three vector fields

$$\mathbf{F} = x\,\mathbf{j}, \qquad \mathbf{F} = -y\,\mathbf{i}, \qquad \mathbf{F} = \frac{1}{2}(-y\,\mathbf{i} + x\,\mathbf{j}),$$

we have $(\partial F_2/\partial x) - (\partial F_1/\partial y) = 1$.

If $\mathcal{C}$ is a positively oriented, piecewise smooth, simple closed curve bounding a region $R$ in the plane, then by Green's Theorem,

$$\oint_{\mathcal{C}} x\,dy = -\oint_{\mathcal{C}} y\,dx = \frac{1}{2}\oint_{\mathcal{C}} (x\,dy - y\,dx) = \iint_R 1\,dA = \text{area of } R.$$

# Green’s Theorem in the Plane

**EXAMPLE**

Use the result of the previous example to calculate the area of the elliptic disk bounded by the curve

$$\mathbf{r} = 3(\cos t + \sin t)\,\mathbf{i} + 2(\sin t - \cos t)\,\mathbf{j}, \qquad 0 \le t \le 2\pi.$$

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**Solution:** The parametrization of $\mathcal{C}$ gives

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$$x\,dy - y\,dx = 6\Big((\cos t + \sin t)^2 + (\sin t - \cos t)^2\Big)dt = 12\,dt.$$

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Thus, by the third formula for the area given in the previous example, the disk has

$$\text{area} = \frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} 12\,dt = 12\pi \text{ square units.}$$

# Green's Theorem in the Plane

**EXAMPLE**

Evaluate

$$I = \oint_{\mathcal{C}} (x - y^3)\,dx + (y^3 + x^3)\,dy,$$

where $\mathcal{C}$ is the positively oriented boundary of the quarter-disk

$$Q: \quad 0 \le x^2 + y^2 \le a^2, \quad x \ge 0,\ y \ge 0.$$

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**Solution:** We use Green's Theorem:

$$\begin{aligned} I &= \iint_Q \left( \frac{\partial}{\partial x}(y^3 + x^3) - \frac{\partial}{\partial y}(x - y^3) \right) dA \\ &= 3 \iint_Q (x^2 + y^2)\,dA = 3 \int_0^{\pi/2} \int_0^a r^3\,dr\,d\theta = \frac{3}{8}\pi a^4. \end{aligned}$$

# Green's Theorem in the Plane

**EXAMPLE**

Let $\mathcal{C}$ be a positively oriented, simple closed curve in the $xy$-plane, bounding a region $R$ and not passing through the origin. Show that

$$\oint_{\mathcal{C}} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \begin{cases} 0, & \text{if the origin is outside } R, \\ 2\pi, & \text{if the origin is inside } R. \end{cases}$$

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**Solution:** First, if $(x, y) \neq (0, 0)$, then by direct calculation,

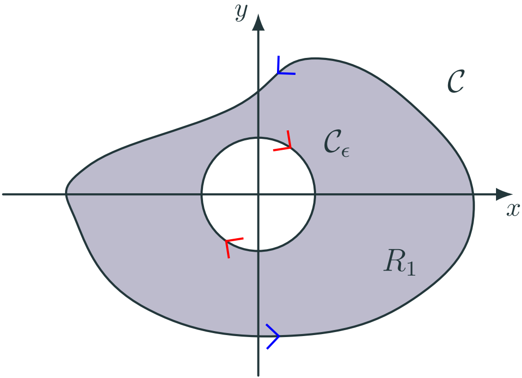
$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = 0.$$

If the origin is not in $R$, then Green's Theorem implies that

$$\oint_{\mathcal{C}} \frac{-y\,dx + x\,dy}{x^2+y^2} = \iint_R \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right] dx\,dy = 0.$$

# Green's Theorem in the Plane

**Solution:** Now suppose the origin is in $R$. Since it is assumed that the origin is not on $\mathcal{C}$, it must be an interior point of $R$. The interior of $R$ is open, so there exists $\varepsilon > 0$ such that the circle $\mathcal{C}_\varepsilon$ of radius $\varepsilon$ centred at the origin is in the interior of $R$. Let $\mathcal{C}_\varepsilon$ be oriented negatively (clockwise).



# Green's Theorem in the Plane

**Solution:** By direct calculation,

$$\oint_{\mathcal{C}_\varepsilon} \frac{-y\,dx + x\,dy}{x^2 + y^2} = -2\pi$$

(because $\mathcal{C}_\varepsilon$ is oriented clockwise).

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(because $\mathcal{C}_\varepsilon$ is oriented clockwise).

Together $\mathcal{C}$ and $\mathcal{C}_\varepsilon$ form the positively oriented boundary of a region $R_1$ that excludes the origin. So, by Green's Theorem,

$$\oint_{\mathcal{C}} \frac{-y\,dx + x\,dy}{x^2 + y^2} + \oint_{\mathcal{C}_\varepsilon} \frac{-y\,dx + x\,dy}{x^2 + y^2} = 0.$$

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Then the desired result follows:

$$\oint_{\mathcal{C}} \frac{-y\,dx + x\,dy}{x^2 + y^2} = -\oint_{\mathcal{C}_\varepsilon} \frac{-y\,dx + x\,dy}{x^2 + y^2} = -(-2\pi) = 2\pi.$$